Conditional Lagrangian acceleration statistics in turbulent flows with Gaussian-distributed velocities

A. K. Aringazin*
Department of Theoretical Physics, Institute for Basic Research, Eurasian National University, Astana 473021, Kazakhstan
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The random intensity of noise approach to the one-dimensional Laval-Dubrulle-Nazarenko-type model having deductive support from the three-dimensional Navier-Stokes equation is used to describe Lagrangian acceleration statistics of a fluid particle in developed turbulent flows. Intensity of additive noise and cross correlation between multiplicative and additive noises entering a nonlinear Langevin equation are assumed to depend on random velocity fluctuations in an exponential way. We use an exact analytic result for the acceleration probability density function obtained as a stationary solution of the associated Fokker-Planck equation. We give a complete quantitative description of the available experimental data on conditional and unconditional acceleration statistics within the framework of a single model with a single set of fit parameters. The acceleration distribution and variance conditioned on Lagrangian velocity fluctuations and the marginal distribution calculated by using independent Gaussian velocity statistics are found to be in a good agreement with the recent high-Reynolds-number Lagrangian experimental data. The fitted conditional mean acceleration is very small, that is, in agreement with direct numerical simulations, and increases for higher velocities but it departs from the experimental data, which exhibit anisotropy of the studied flow.

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I. INTRODUCTION

Data analysis and modeling of statistical properties of a Lagrangian particle advected by a fully developed turbulent flow is of much practical interest and complements traditional studies made in the Eulerian framework. The strong and nonlocal character of Lagrangian particle coupling due to pressure effects makes the main obstacle to derive turbulence statistics from the Navier-Stokes equation. Recent breakthrough Lagrangian experiments [1–3] have motivated growing interest to a single-particle statistics. Some phenomenological approaches [4,5] inspired by the nonextensive statistics formalism were used [6] to describe Lagrangian acceleration of a fluid particle in a stationary developed turbulent flow within the framework of a Langevin type equation; see also Refs. [7–9].

Some toy models of developed turbulence suffer from the lack of justification of a fit from turbulence dynamics [14], and the connection between specific nonthermodynamical processes and nonextensive mechanisms was argued to be generally not well defined [15]. Recent one-dimensional (1D) stochastic particle models and their refinements [10–12] were reviewed in Ref. [13], in which the importance of Navier-Stokes equation based approaches is emphasized.

Fluid particle dynamics in a developed turbulent flow is described in terms of a generalized Brownian motion with the Lagrangian acceleration of an individual particle viewed as a dynamical variable. In the data processing, the acceleration is associated with the Lagrangian velocity increment in time for sufficiently small time scales, in a far dissipative subrange where turbulent fluctuations are smoothed. Such models are generally based upon a hierarchy of characteristic time scales in the system and naturally employ a one-point statistical description using a Langevin type equation, or the associated Fokker-Planck equation for a one-point probability density function of the variable. Noises entering the Langevin type equation are treated along a fluid particle trajectory, and the Fokker-Planck approximation makes connection between the dynamics and statistical approach.

Recently we have shown [13,16–18] that the 1D Laval-Dubrulle-Nazarenko (LDN) toy model [19,20] of the acceleration evolution with the model turbulent viscosity ν_t and coupled δ-correlated Gaussian multiplicative and additive noises is in a good agreement with the high-precision Lagrangian experimental data on acceleration statistics [1–3]; the Taylor microscale Reynolds number $R_\lambda=690$, the measured normalized acceleration range is $-60\leqslant a/\langle a^2\rangle^{1/2}\leqslant 60$, and the resolution is 1/65 of the Kolmogorov length scale of the flow. The long-standing Heisenberg-Yaglom scaling of a component of Lagrangian acceleration, $\langle a^2\rangle = a_0\overline{u}^{9/2}\nu^{-1/2}L^{-3/2}$, was confirmed experimentally [1] to a very high accuracy, for about seven orders of magnitude in the acceleration variance, or two orders of the root mean square velocity $\overline{u}$, at $R_\lambda>500$; a_0 is the Kolmogorov constant, ν is the kinematic viscosity, and L is the Lagrangian integral length scale. Long-time correlations and the occurrence of very large fluctuations at small scales dominate the motion of a fluid particle, and this leads to a new dynamical picture of turbulence [21,22].

The original 3D and 1D LDN models were formulated both in the Lagrangian and Eulerian frameworks for small-scale velocity increments in time and space, respectively. They are based on the Gabor transformation (Fourier transform in windows) and a stochastic kind of Batchelor-Proudman rapid distortion theory (RDT) approach to the in-

*Also at Department of Mechanics and Mathematics, Kazakhstan Division, Moscow State University, Moscow 119899, Russia. Electronic address: aringazin@mail.kz

compressible 3D Navier-Stokes equation [19], and thus have a deductive support from turbulence dynamics. A study based on direct numerical simulations (DNS's) of the 3D LDN model in the regime of decaying turbulence has been made.

The random intensity of noise (RIN) approach [13,16] provides an extension of the 1D LDN model viewed in the limit of small time-scale τ for which Lagrangian velocity increments are proportional to τ: $u(t+\tau)-u(t)=\tau a(t)$.

The main idea of the RIN approach is simply to account for the recently established two well separated Lagrangian autocorrelation time scales for the velocity increments [21] and assume that certain model parameters, such as intensity of noise and fluctuate at *large* time scale.

An analysis of such a simple 1D model can shed some light to properties of the 3D LDN model of Lagrangian dynamics. Recent development of the 3D LDN model can be found in Ref. [23], in which some interesting methods of turbulent dynamo problem have been exploited.

The experimental data on the axial component of Lagrangian acceleration a of polystyrene tracer particle in the $R_\lambda=690$ water flow generated between counter-rotating disks have been fitted by the stretched exponential [1–3],

$$P(a) = C \exp\left[- \frac{a^2}{(1 + |b_1 a/b_2|^{b_3}) b_2^2} \right]. \qquad (1)$$

Here, $b_1=0.513\pm0.003$, $b_2=0.563\pm0.02$, and $b_3=1.600\pm0.003$ are fit parameters, and $C=0.733$ is a normalization constant. The studied flow is highly anisotropic at large scales and this appears to affect small scales, resulting in a small skewness of the acceleration distribution and observable distinction in the distributions of different components of the velocity. At large acceleration values, tails of the distribution (1) decay asymptotically as $\exp[-|a|^{0.4}]$ that implies a convergence of the fourth-order moment $\langle a^4\rangle=\int_{-\infty}^{\infty} a^4 P(a)da$, as confirmed by the experiment with measured normalized acceleration values up to $|a|=60$. The flatness factor of the distribution (1) which characterizes widening of its tails (when compared to a Gaussian) is $F\equiv\langle a^4\rangle/\langle a^2\rangle^2=55.1$, that is in agreement the experimental value $F=55\pm8$. The Kolmogorov time of the flow is $\tau_\eta=0.93$ ms. Low-pass filtering with the $0.23\tau_\eta$ width of the collected 1.7×10^8 data points was used, and the response time of the optically tracked 46-μm tracer particle is $0.12\tau_\eta$.

Recently, Chevillard *et al.* [22] have constructed an appropriately recasted multifractal approach to describe statistics of Lagrangian velocity increments in a wide range of time scales, from the integral to dissipative one. The resulting theoretical distribution reproduces continuous widening of the velocity increment probability density function (PDF) with the decrease of time scale, from a Gaussian-shaped to the stretched exponential as observed in Lagrangian experiments carried out at Cornell [1–3] and ENS-Lyon [21,24], and DNS of the 3D Navier-Stokes equation. Two global parameters (Reynolds number and Lagrangian integral time scale) and two local parameters (intermittency parameter and smoothing parameter) with a parabolic singularity spectrum were used to cover the data in the entire range of time scales. At dissipative time scale the obtained PDF fits the experimental data on Lagrangian acceleration to a good accuracy. The cumulant analysis made in this approach provides an understanding of the observed departures from the scaling when going from the integral to dissipative timescale. The used parabolic singularity spectrum $D(h)$ is a hallmark of the log-normal (Kolmogorov, 1962) statistics and reproduces well the left-hand side (corresponding to intense velocity increments) of the observed curve which is centered at $0.58(>1/2)$ but increasingly deviates at the right-hand side of it (corresponding to weak velocity increments). Another widely used statistics, the log-Poisson one, was shown to depart from Lagrangian observations in the same manner. The conditional acceleration statistics was not considered in this work.

In a recent paper A. Reynolds [25] developed a self-consistent second-order stochastic model with additive noise which accounts for dependence of the Lagrangian acceleration covariance matrix on Lagrangian velocities u. The observed dependence of the conditional acceleration variance $\langle a^2|u\rangle$ on u [3] was partially understood in terms of Lagrangian accelerations induced by vortex tubes within which the vorticity is constant and outside which the vorticity vanishes. Scaling relations were invoked to derive a third-order polynomial structure of the isotropic covariance matrix as a function of squared velocity u^2 [26]. The inclusion of such conditional acceleration covariances in the model resulted in reduction of the predicted occurrence of small accelerations that meets experimental and DNS data for unconditional distributions. The cores of the resulting conditional acceleration distributions were found to broaden with increasing u, in a qualitative agreement with the experiment.

Sawford *et al.* [26] have studied acceleration statistics from laboratory measurements and direct numerical simulations in 3D turbulence at R_λ ranging from 38 to 1000. For large $|u|$, the conditional acceleration covariance behaves like u^6. This is qualitatively consistent with the stretched exponential tails of the unconditional acceleration PDF. The conditional mean rate of change of the acceleration derived from the data has been shown consistent with the drift term in second-order Lagrangian stochastic models of turbulent transport. The correlation between the square of the acceleration and the square of the velocity has been shown small but not negligible.

In very recent papers Biferale *et al.* [27,28] have presented interesting results of DNS of Lagrangian transport in homogeneous and isotropic turbulence with R_λ up to 280, a very accurate resolution of dissipative scales, and an integration time of about Lagrangian time scale. They have shown how the multifractal formalism offers an alternative approach which is rooted in the phenomenology of turbulence. The Lagrangian statistics was derived from the Eulerian statistics without introducing *ad hoc* hypotheses. Although the formalism is not capable to account for small acceleration values (typical situation for the multifractal approach), the obtained acceleration PDF captures the DNS data well in the tails, with normalized acceleration values ranging from about $|a|/\langle a^2\rangle^{1/2}=1$ up to $|a|/\langle a^2\rangle^{1/2}=80$. Alas, one can observe an overestimation in this range which can be clearly seen from the predicted contribution to the fourth-order moment, $a^4P(a)$, as compared to the DNS data. The high degree of isotropy of the simulated stationary flow suggests equivalence of Cartesian components of acceleration aligned to fixed directions, and the resulting DNS distribution obtained by averaging over the components has been found with no observable asymmetry with respect to $a \to -a$. The multifractal approach has been also used [28] to obtain acceleration moments conditional on the velocity. Particularly, the multifractal prediction $\langle a^2|u\rangle \sim u^{4.57}$ agrees well with the DNS data for large velocity magnitudes. The predicted exponent 4.57 differs from the value 6 predicted recently by Sawford *et al.* [26] and is very close to the Heisenberg-Yaglom scaling exponent value 9/2. This indicates that the averaging of the above conditional acceleration variance $\langle a^2|u\rangle$ over Gaussian distributed velocity u is consistent with the Heisenberg-Yaglom scaling law [see remark and Eq. (68) of Ref. [13]].

In the present paper, we focus on the 1D LDN type dynamical modeling of the Lagrangian acceleration *conditional on velocity fluctuations* presented recently by Mordant, Crawford, and Bodenschatz in the experimental work [3]. In contrast to our previous studies of the conditional acceleration statistics [13,16–18], here we give a self-consistent treatment of the model by explicit accounting for a Gaussian distribution of Lagrangian velocity fluctuations that is observed experimentally. We give a complete quantitative description of the available experimental data on *both* the conditional and unconditional acceleration statistics within the framework of a single model with a single set of fit parameters. The importance of the present approach is that the Lagrangian single-particle modeling is dynamical and has a deductive support from the Navier-Stokes equation, with few assumptions justified by the turbulence phenomenology being used. This approach adds a different look to homogeneous isotropic turbulence modeling which is alternative to those given by the recent multifractal and *ad hoc* Langevin stochastic approaches.

It should be emphasized that the Lagrangian velocity is known to follow Gaussian distribution to a very good accuracy while the Lagrangian acceleration follows highly non-Gaussian distribution which is related to extremely intermittent character of the acceleration, with pronounced central peak and relatively frequent acceleration bursts up to 80 standard deviations. We note that, theoretically, time derivative of a dynamical variable does not necessarily follow the same statistical distribution as that of the variable.

Our consideration is restricted to a stationary one-point distribution function. Two-point statistical analysis is of much interested and can be made elsewhere.

The paper is organized as follows. In Sec. II we give a brief description of the 1D LDN model and present the resulting acceleration distribution, which we treat as a conditional one by assigning stochastic properties to certain parameters. In Sec. III we briefly review results of our previous work and make sample fits of the obtained conditional and unconditional acceleration distributions and moments to the experimental data. In Sec. IV we discuss the obtained results and make conclusions.

II. 1D LAVAL-DUBRULLE-NAZARENKO MODEL OF SMALL-SCALE TURBULENCE

In this section, we present only a brief sketch of the 1D LDN model and refer the reader to Refs. [13,19] for more details; see also Ref. [23]. This toy model can also be viewed as a passive scalar in a compressible 1D flow.

The main assumption of the LDN approach to the 3D Navier-Stokes turbulence is to introduce and separate large-scale and small-scale parts in the 3D Navier-Stokes equation by using the Gabor transformation [19]. This allows us to consider analytically small-scale turbulence coupled to large-scale terms (the interscale coupling). The approach allows one to account for nonlocal interactions which were argued to be important in understanding intermittency in developed turbulent flows. The other, large-scale, part of the equation can be treated separately (and, in principle, solved numerically given the forcing and boundary conditions) since the forcing is characterized by presumably narrow range of small wave numbers, and the small scales make little effect on it. Small-scale interactions are modeled by a turbulent viscosity and were shown numerically to make a small contribution to the anomalous scaling (intermittency) in the decaying turbulence. Nevertheless, these are important when fitting model distribution to the experimental data. The 3D LDN model of small scale turbulence was used to formulate a simplified 1D LDN model, which was studied both in the Eulerian and Lagrangian frames [19].

We use probability density function obtained as a stationary solution of the Fokker-Planck equation that corresponds to a consideration of statistically stationary state; statistical homogeneity and isotropy of the 3D flow is assumed as well. This equation is derived from the Langevin equation for a component of Lagrangian acceleration $a(t)$ [13,19],

$$\frac{\partial a}{\partial t} = (\xi - \nu_t k^2)a + \sigma_\perp, \tag{2}$$

where $\nu_t = \sqrt{\nu_0^2 + B^2a^2/k^2}$ is the turbulent viscosity modeling small-scale interactions, ν_0 is constant kinematic viscosity, B is free parameter measuring the contribution of nonlinearity in a to the turbulent viscosity, and k is wave number; $\partial_t k = -k\xi$, $k(0) = k_0$, to model the RDT stretching effect in the one-dimensional case.

In the original 3D LDN model based on the Navier-Stokes equation, $\xi(t)$ is related to the velocity derivative tensor and $\sigma_\perp(t)$ describes a forcing of small scales by large scales via the energy cascade mechanism (nonlocal interscale coupling). In the 1D LDN model, these are approximated by a sufficiently simple statistics inspired by the Kraichnan ensemble used for turbulent passive scalar and the Kraichnan-Kazantsev model of turbulent dynamo: external Gaussian white-in-time noises along a fluid particle trajectory,

$$\langle \xi(t)\rangle = 0, \quad \langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \quad \langle \sigma_\perp(t)\rangle = 0,$$

$$\langle \sigma_\perp(t)\sigma_\perp(t')\rangle = 2\alpha\delta(t-t'), \quad \langle \xi(t)\sigma_\perp(t')\rangle = 2\lambda\delta(t-t'). \tag{3}$$

Here, D, α, and λ are free parameters measuring intensity of the noises and their cross correlation, respectively. Bigger D

and α means bigger contribution of the velocity derivative tensor and the interscale coupling (both viewed here as short-time autocorrelated processes) to the small-scale dynamics.

The model (3) imposes an obvious limitation but is partially justified by DNS in the laboratory frame of reference [19]. The averaging is made over ensemble realizations. Zero means correspond to isotropy of the forces. Physically, the small scales are thus assumed to be stochastically distorted by much larger scales. We stress that a correlation between the noises ξ and $\sigma_\perp$ is not *ad hoc* assumption but a consequence of their structure as they contain the same large-scale velocity serving as a unifying agent between the noises.

It should be emphasized that the 1D LDN toy model and its particular case (2) have several limitations related to the LDN separation of small and large scales allowing to study exclusively nonlocal effects associated to the linear process of distortions of small scales by a strain produced by large scales, the use of model turbulent viscosity, and one dimensionality.

In the Lagrangian frame the wave number k is replaced in terms of the initial value $k_0=k(0)$ and time while the parameters acquire dependence on k_0 [19]; we drop the subscript 0 in k_0 in Eq. (2) and subsequent formulas to simplify notation.

Thus one makes a closure by treating the combined effect of large scales, for which one has a different dynamical LDN equation that could be in principle solved numerically [23], and nonlocal interscale coupling, as a pair of given external noises. The large-scale dynamics is local in wave number space and hence it is weakly affected by the small scales. The price of the simplification (3) is that one introduces free parameters to the description. Matching small-scale dynamics to the large-scale one deserves a separate study. Despite that 3D turbulence is known to be more sensitive to large-scale forcing or boundary conditions, as compared to the 2D one, the used simplification (3) is relevant for high-Reynolds-number flows to some extent [19,23], and allows one to advance in the analytical treatment of the problem. It should be noted that scaling properties of the system described by Eq. (4) reveal a robust character with respect to the selection of noises ξ and $\sigma_\perp$ (see Ref. [13] and references therein).

The acceleration PDF stemming from the stochastic model (2) and (3) has been calculated exactly in our previous work [13],

$$P(a) = C\exp\left[\int_0^a dx\frac{-k^2x\sqrt{\nu_0^2+B^2x^2/k^2}-Dx+\lambda}{Dx^2-2\lambda x+\alpha}\right] = \frac{C\exp[-\nu_t k^2/D+F(c)+F(-c)]}{(Da^2-2\lambda a+\alpha)^{1/2}(2Bka+\nu_t k^2)^{2B\lambda k/D^2}}, \quad (4)$$

for constant parameters. Here, C is normalization constant and we have denoted

$$F(c) = \frac{c_1k^2}{2c_2D^2c}\ln\left(\frac{2D^3}{c_1c_2(c-Da+\lambda)}\times[B^2(\lambda^2+c\lambda-D\alpha)a+c(D\nu_t^2k^2+c_2\nu_t)]\right), \quad (5)$$

$$c = -i\sqrt{D\alpha-\lambda^2}, \quad (6)$$

$$c_1 = B^2(4\lambda^3+4c\lambda^2-3D\alpha\lambda-cD\alpha)+D^2(c+\lambda)\nu_0^2k^2, \quad (7)$$

$$c_2 = \sqrt{B^2(2\lambda^2+2c\lambda-D\alpha)k^2+D^2\nu_0^2k^4}. \quad (8)$$

The distribution (4) is characterized by the presence of exponential cut off, complicated power-law dependence, and terms responsible for a skewness (asymmetry with respect to $a\to -a$).

One way when comparing the model with the experiment is to make a direct fit of the obtained PDF (4) to the experimental data on unconditional acceleration distribution by assuming all the parameters and wave number to be constant.

Particularly, this implies a reduction of the original 1D LDN model since the wave number is taken to be fixed so that the artificial 1D compressibility aimed to model the RDT stretching effect in the 1D case is not considered. We note that the Lagrangian acceleration is usually associated to the dissipative scale, and in the present paper we do not study dependence of the parameters on the wave number. Such a dependence for velocity increments was analyzed in Ref. [19] with the expected result that for larger scales the velocity increment PDF tends to a Gaussian form. The Gaussian form is reproduced also when $D\to 0$ and $B\to 0$, i.e., the process becomes purely additive with a linear drift term.

Without loss of generality one can put, in a numerical study, $k=1$ and the additive noise intensity $\alpha=1$ by rescaling the multiplicative noise intensity $D>0$, the turbulent viscosity parameter $B>0$, the kinematic viscosity $\nu_0>0$, and the cross correlation parameter λ. The particular cases $B=0$ and $\nu_0=0$ at $\lambda=0$, and the general case at $\lambda=0$ were studied in detail in Ref. [13]. Nonzero λ is responsible for an asymmetry of the PDF (4) and in 3D picture corresponds to a correlation between stretching and vorticity (the energy cascade). Particularly, in the Eulerian framework the third-order moment of spatial velocity increment $\langle(\delta_l u)^3\rangle$ was found to be proportional to the cross-correlation parameter, in accord to a kind of generalized Kármán-Howarth relationship [19]. However, the approximation based on constant parameters does not allow one to consider both the conditional and unconditional acceleration statistics.

In the next section, we extend the model (2)–(4) by assuming certain model parameters in Eq. (4) to be dependent on random velocity fluctuations. This extension is compatible with the 3D LDN approach as ξ and $\sigma_\perp$ depend on velocity fluctuations and contain large-scale quantities due to their definitions [19]. Such a functional dependence and long-time fluctuations have been ignored when making the simplification (3). We partially restore them. This is the main point of our consideration, and the functional form of the distribution is thus due to Eq. (4) with certain parameters being now treated as functions of stochastic velocity u. Observations are that the acceleration variance does depend on the same component of velocity fluctuations. Local homogeneity assumed by the Kolmogorov 1941 theory is thus broken that is a prerequisite to describe turbulence intermittency. The scaling approach indicates an essential character

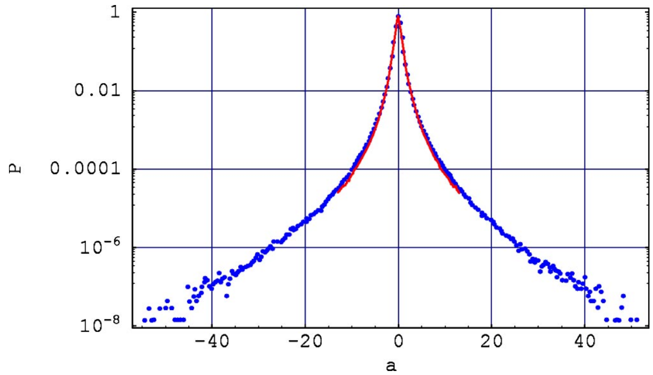

FIG. 1. A comparison of the experimental unconditional Lagrangian acceleration PDF (dots) and the experimental conditional Lagrangian acceleration PDF at velocity fluctuations $u=0$ (line) [2,3]; the acceleration component a is normalized to unit variance.

of such a dependence. Lagrangian intermittency is known to be much stronger than the Eulerian one due to existence of very intense vortical structures at small scales and absence of the so-called sweeping effect in the Lagrangian frame.

We point out that the characteristic time of variation of the parameters should be sufficiently large to justify approximation that the resulting PDF (4) is used with independent randomized parameters, $P(a|\text{parameters})$. Two well separated time scales in the Lagrangian velocity increment autocorrelation have been established both by experiments and DNS [21]. The large time scale has been found of the order of the Lagrangian integral scale and corresponds to a magnitude part that is in accord to our assumption that the intensity of noise along the trajectory is long-time fluctuating.

III. CONDITIONAL ACCELERATION STATISTICS

The experimental unconditional and conditional distributions, which we denote for brevity by $P_{\text{expt}}(a)$ and $P_{\text{expt}}(a|u)$, respectively, were found to be approximately of the same stretched exponential form at $u=0$ (Fig. 1), and both reveal a strong Lagrangian turbulence intermittency [3]. This similarity indicates that they share the same process underlying the intermittency.

Accordingly, in our previous studies [13,16–18] we used the result of our direct fit of the PDF (4) to $P_{\text{expt}}(a)$, which was measured with a high precision; 3% relative uncertainty for $|a|/\langle a^2\rangle^{1/2}\leqslant 10$ [2,3]. We assumed that the parameters α and λ entering Eq. (4) depend on the amplitude of Lagrangian velocity fluctuations u, while D, B, and ν_0 are taken to be fixed at the fitted values ($k=1$). Theoretically, only α and λ depend explicitly on large-scale velocity due to 3D LDN model, while the other parameters do not.

An exponential form of $\alpha(u)$ has been proposed in Ref. [13] and was found to be relevant from both the (Kolmogorov, 1962) phenomenological and experimental points of view. Particularly, such a form leads to the log-normal RIN model when u is independent Gaussian distributed with zero mean [12], and yields the acceleration PDF whose low-probability tails are in agreement with experiments [6,13]. Also, we used an exponential form of $\lambda(u)$ so that the conditional acceleration PDF (4) takes the form $P(a|u) = \boldsymbol{P}(a|\alpha(u),\lambda(u))$. Such a form was found to provide good fits of (i) the conditional probability density function $P(a|u)$ to $P_{\text{exp}}(a|u)$; (ii) the conditional acceleration variance $\langle a^2|u\rangle$; and (iii) the conditional mean acceleration $\langle a|u\rangle$ [17] at various u that meet the experimental data [3]. A brief report on these results is presented in Ref. [18].

However, a self-consistent consideration of the model assumes fitting of $P(a|u)$ to $P_{\text{expt}}(a|u)$, and the marginal PDF computed due to

$$P_{\text{m}}(a) = \int_{-\infty}^{\infty} P(a|u)g(u)du, \tag{9}$$

where $g(u)$ is PDF of independent velocity fluctuations, should reproduce $P_{\text{expt}}(a)$. The marginal distribution corresponds to a convolution of the stationary acceleration statistics with independent random velocity fluctuations.

In the present paper, we fill this gap. Our task is to fit a variety of the experimental data, both on the conditional and unconditional statistics of acceleration, with a single set of fit parameters. For this purpose we use the following natural steps.

First we fit $\boldsymbol{P}(a|u)=P(a|\alpha(u),\lambda(u))$ given by Eq. (4) to $P_{\text{expt}}(a|u)$ [3] assuming that the parameters depend on u in an exponential way,

$$\alpha(u) = \alpha_0 \exp[|u|/u_\alpha], \quad \lambda(u) = \lambda_0 \exp[|u|/u_\lambda]. \tag{10}$$

Hereafter, we use normalized acceleration a and velocity fluctuations u. The fit parameter set is $D>0$, $\nu_0>0$, $B>0$, λ_0, $u_\alpha>0$, and $u_\lambda>0$ ($\alpha_0=1$, $k=1$). The relations in Eq. (10) mean that the additive noise intensity and the correlation between the noises become higher for bigger velocity fluctuations $|u|$.

We fit $P(a|0)$ to $P_{\text{expt}}(a|0)$, that excludes u_α and u_λ from consideration, by varying D, ν_0, and B at $\alpha_0=1$ and $\lambda_0=-0.005$. We notice that the available conditional statistics $P_{\text{expt}}(a|u)$ is low for high velocities, the presented acceleration range is small, $-14<a<14$, so that a rather big uncertainty remains when determining fit values of the parameters. Changes in shape of $P_{\text{ext}}(a|u)$ with u increasing from $u=0$ to $u=3.1$ are captured independently by the fit parameters u_α and u_λ. The result is shown in Fig. 2. Good overlapping of each curve with data points at all fixed magnitudes of u has been achieved.

Second we calculate the conditional mean $\langle a|u\rangle$ and the conditional variance $\langle a^2|u\rangle$ and compare them with the experimental data. This decreases uncertainty in fit parameter values. The results are shown in Figs. 3 and 4. Note that $\langle a|u\rangle$ as a function of u is very small that does not match the experiment. We will discuss this in Sec. IV below.

Finally we calculate numerically the marginal distribution $P_{\text{m}}(a)$ given by Eq. (9) with the conditional PDF $\boldsymbol{P}(a|\alpha(u),\lambda(u))$ and Gaussian distribution of velocity fluctuations,

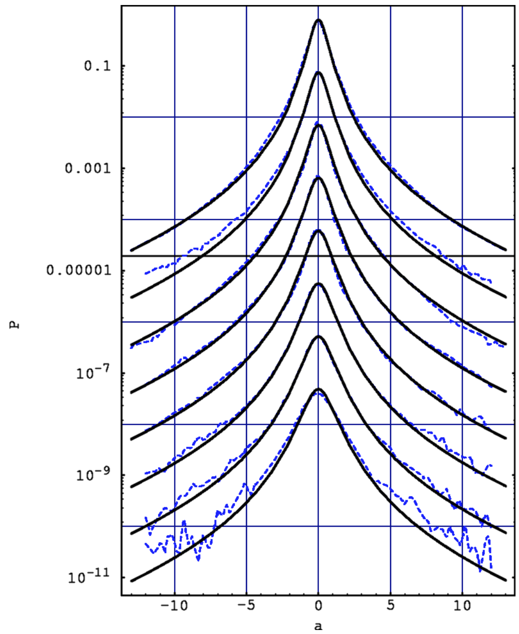

FIG. 2. Theoretical conditional acceleration PDF $P(a|u)$ (line) and the experimental conditional acceleration PDF (dashed line) at velocity fluctuations $u=0,0.45,0.89,1.3,1.8,2.2,2.7,3.1$ [3] (from top to bottom, shifted by repeated factor 0.1 for clarity); the acceleration component a is normalized to unit variance, and the same component of velocity u is given in root mean square units.

$$g(u) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{u^2}{2}\right], \qquad (11)$$

at fixed a ranging from −100 to 100 with the step 0.1. Then we make an interpolation and fit it to $P_{\rm expt}(a)$. A noticeable effect of the integration over u with Gaussian $g(u)$ is a widening of tails of the distribution that meets Fig. 1; the integration range $-20 \leqslant u \leqslant 20$ has been used. The fit of $P_{\rm m}(a)$ to $P_{\rm expt}(a)$ strongly decreases the uncertainty but the most strict determination of fit values comes due to a comparison of the theoretical contribution to fourth-order moment, $a^4P(a)$, with the experimental data. The results are shown in Figs. 5–7.

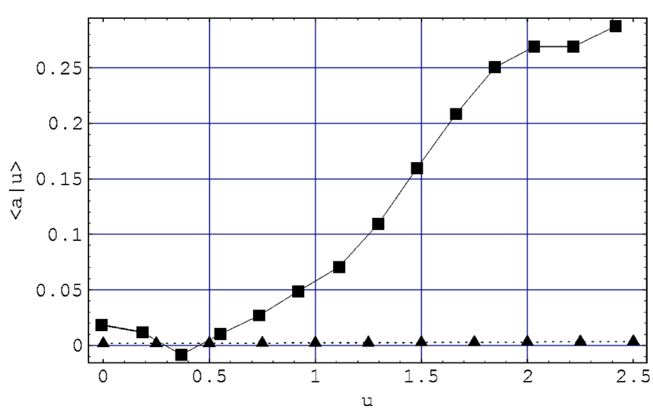

FIG. 3. Theoretical conditional acceleration mean $\langle a|u\rangle$ (triangles) and the experimental conditional acceleration mean (squares) as functions of velocity fluctuations.

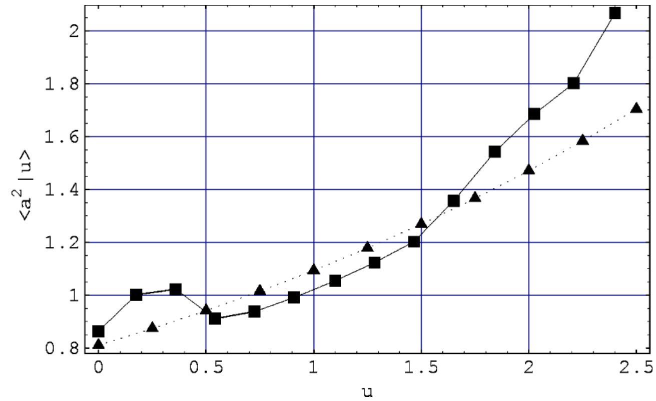

FIG. 4. Theoretical conditional acceleration variance $\langle a^2|u\rangle$ (triangles) and the experimental conditional acceleration variance (squares) as functions of velocity fluctuations.

The quality of these sample fits is better than in the other recent stochastic models reviewed in Ref. [13]. In particular, the core of the unconditional distribution reproduces very well that given by the stretched exponential (1) as shown in Fig. 6. However, both curves a bit underestimate the height at $a=0$.

The value $\lambda_0=-0.005$ has been obtained by adjusting the theoretical curve to slightly different heights of the peaks of the observed $a^4P(a)$ shown in Fig. 7. Note that the model does not assume the use of *ad hoc* skewness of the forcing. Nonzero cross correlation parameter λ naturally results not only in small mean acceleration but also in a skewness of both the theoretical distributions $P(a|u)$ and $P_{\rm m}(a)$. This skewness may be associated to the Eulerian downscale skewness generation, which despite of being small for homogeneous flows is known to be of a fundamental character in the inertial range (Kolmogorov four-fifths law), since the Eulerian $\langle(\delta_l u)^3\rangle$ was found to be proportional to cross-correlation parameter.

We stress that the observed very small skewness of acceleration distribution is attributed to the effect of anisotropy of the studied flow. How the large-scale asymmetry affects smallest scales of the flow is an interesting problem. Our fit made by using nonzero λ is of an illustrative character, to verify whether it can explain the observed increase of the

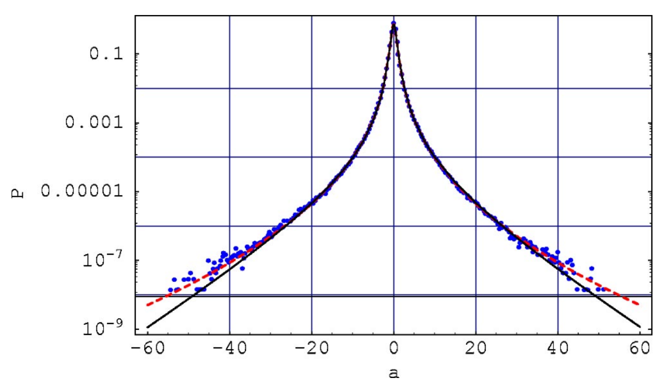

FIG. 5. Theoretical marginal PDF (9) for Gaussian distributed velocities (line), experimental data at $R_\lambda=690$ [2] (dots), and the stretched exponential fit (1) (dashed line).

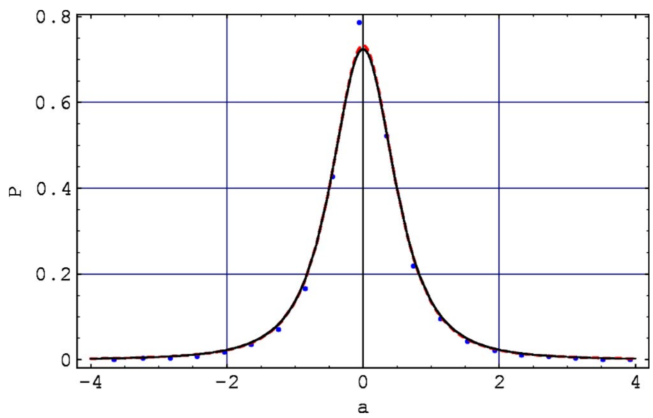

FIG. 6. Double-linear plot of the central part of the curves of Fig. 5. Same notation as in Fig. 5.

conditional mean acceleration with increasing velocity depicted in Fig. 3. This issue will be discussed further in Sec. IV.

The following remarks are in order. Our finding is that the condition $u_\alpha \leqslant u_\lambda$ provides a convergence of $P_m(a)$. Also, u_λ should not be small to provide assumed condition $\lambda \ll \alpha$ at arbitrary u (the cross correlation is small as compared to both noise intensities α and D) [13,19]. We used these criteria when making the fits.

The resulting sample fit values are given by

$$D = 2.1, \quad \nu_0 = 5.0, \quad B = 0.35,$$
$$\lambda_0 = -0.005, \quad u_\alpha = 3.0, \quad u_\lambda = 3.0, \qquad (12)$$

with $\alpha_0 = 1$ and $k = 1$. The theoretical curves in Figs. 2–7 are shown for this sample set of values, which require a further fine tuning. Such a small value of λ as compared to α or D is in agreement with that obtained in the LDN direct numerical simulations. The calculated flatness factor $F = 49.3$ of $P_m(a)$ is in agreement with the experimental value 55 ± 8.

To summarize, the considered Navier-Stokes equation based 1D toy model (4)–(10) is capable to fit all the available high-precision experimental data on the conditional and unconditional Lagrangian acceleration statistics [1–3] with the single set of parameters (12) to a good accuracy, with an exception being only the conditional mean acceleration.

IV. DISCUSSION AND CONCLUSIONS

One can see from Fig. 3 that at the values of fit parameters (12) the predicted conditional mean acceleration $\langle a|u\rangle$ qualitatively is in agreement but does not reproduce the experimental data. Namely, it is nonzero due to nonzero λ and increases with the increase of $|u|$ but remains to be very small even at high values of $|u|$. The conditional mean acceleration is evidently zero for a symmetrical distribution ($\lambda = 0$) and should be zero for statistically homogeneous isotropic turbulence. The observed departure from zero is thought to reflect anisotropy of the studied flow albeit the DNS of homogeneous isotropic turbulence also reveals slightly nonzero mean [3].

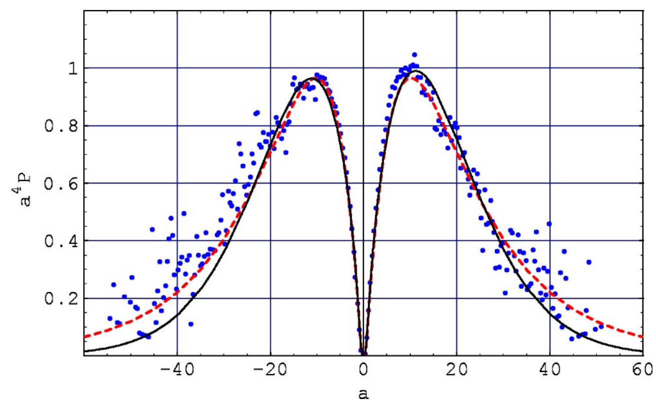

FIG. 7. Contribution to the kurtosis $a^4P(a)$. Same notation as in Fig. 5.

To reduce the discrepancy, we have tried the value $u_\lambda = 1.0$ instead of $u_\lambda = 3.0$ to provide faster increase of $|\lambda|$ for higher $|u|$. This implies a good fit to the experimental conditional mean acceleration (see, e.g., Fig. 2 in Ref. [18]) but we found an excess asymmetry of $P(a|u)$ at high u, with big departure from observations, and divergencies when calculating $P_m(a)$. The reason of the divergency is that $\lambda(u)$ at $u_\lambda = 1.0$ grows faster than $\alpha(u)$ at $u_\alpha = 3.0$ so that λ becomes comparable or bigger than α with increasing u, and when $\lambda^2 \to D\alpha$ the function $F(c)$ defined by Eq. (5) undergoes unbound growth. Thus we conclude that the observed conditional mean acceleration is mainly due to the flow anisotropy effect rather than some intrinsic dynamical mechanism associated to the developed turbulence.

In general one observes a rather small relative increase of the conditional mean acceleration for higher $|u|$ that eventually reflects a coupling of the acceleration to large scales of the studied flow [19,22]. This coupling could be accounted also by introducing a correlation between the acceleration and velocity fluctuations. This possibility is of much interest to explore as it may yield the deficient increase of $\langle a|u\rangle$ but it is beyond the scope of the present formalism, which assumes an independent velocity statistics. We note also that in contrast to the experimental data on the variance $\langle a^2|u\rangle$ the experimental $\langle a|u\rangle$ exhibits small asymmetry with respect to $u \to -u$ (not shown in Fig. 3).

In the present paper, the multiplicative noise intensity D was taken to be independent on the velocity fluctuations u. The effect of variation of D has been considered in Ref. [13] with the qualitative result that it does not provide the specific change in shape of $P(a|u)$ observed in experiments. However, a weak dependence of D on u cannot be ruled out.

In summary, the presented 1D LDN type stochastic toy model with the velocity-dependent additive noise intensity and cross correlation parameter is shown to capture the main features of the observed conditional and unconditional Lagrangian acceleration statistics to a good accuracy except for the discrepancy in the conditional mean acceleration which can be attributed to certain coupling of the acceleration to large scales of the studied flow.

The main result is of course not only good sample fits which are important to test the performance of the model but also certain advance in understanding of the mechanism of Lagrangian intermittency provided by the dynamical Laval-Dubrulle-Nazarenko approach to small-scale turbulence.

The central point is that the LDN toy model has a strong deductive support from the Navier-Stokes turbulence. The obtained exact analytic result for the conditional acceleration distribution and the use of recent high-precision Lagrangian experimental data on conditional and unconditional acceleration statistics provide a detailed analysis of the mechanism within the adopted framework. Effects of large time scales (nonlocality) and turbulent viscosity (nonlinearity) have been found of much importance in Lagrangian acceleration steady-state statistics. The detailed study of conditional acceleration statistics have revealed a specific model structure of the external large-scale dynamics and nonlocal interscale coupling for homogeneous high-Reynolds-number flows. The additive noise associated to the downscale energy transfer mechanism encodes the main contribution to the velocity dependence of the acceleration statistics. The cross correlation between the model additive and multiplicative noises associated to a correlation between stretching and vorticity naturally provides a skewness of distributions and a nonzero mean. The weakness of this correlation is a theoretical requirement that meets the Lagrangian and Eulerian experiments and DNS of homogeneous isotropic turbulence. The observed conditional mean acceleration is mainly related to the flow anisotropy. The cross correlation is related to the four-fifths Kolmogorov law but the effect of skewness is negligibly small as the result of relatively large intensity of the additive noise, which tends to symmetrize acceleration distributions. This is a dynamical evidence implied by the model rather than a direct consequence of *a priori* assumption on isotropy in the spirit of the Kolmogorov 1941 theory. The use of exponential dependence of certain noise parameters on statistically independent Gaussian distributed Lagrangian velocity fluctuations has been found appropriate to cover different experimental data on conditional statistics and to transfer from the conditional to unconditional acceleration distribution both exhibiting a strong Lagrangian intermittency of the flow. Such a dependence is also compatible with the log-normal statistics assumed by the Kolmogorov 1962 theory. The Gaussian white-in-time multiplicative noise and long-time correlated intensity of the additive noise were both found to make an essential contribution to intermittent bursts.

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